J. Fluid Mech. (1982), *vol.* 124, *pp.* 475–494
Printed in Great Britain

Taylor–Gortler vortices in fully developed or boundary-layer flows: linear theory

By P. HALL
Department of Mathematics, Imperial College, London

(Received 5 May 1981)

The stability characteristics of some fluid flows at high Taylor or Görtler numbers are determined using perturbation methods. In particular, the stability characteristics of some fully developed flows between concentric cylinders driven either by a pressure gradient or the motion of the inner cylinder are investigated. The asymptotic structure of short-wavelength disturbances to these flows is obtained and used as a basis for a formal perturbation solution to the corresponding stability problem appropriate to a developing boundary layer. The non-parallel effect of the basic flow on the condition for neutral stability is discussed. The results obtained suggest that the disturbances are concentrated in internal viscous or critical layers well away from the wall and the free stream. The stability of a boundary layer on a concave wall to Görtler vortices that propagate downstream is also considered. These modes are found to be more stable than the usual time-independent modes and they propagate downstream with the speed of the basic flow in the critical layer. Some comparison with previous experimental and theoretical work is given.

1. Introduction

The primary aim of this paper is to show how boundary-layer growth influences the growth of Görtler vortices in developing flows on concave surfaces. The existence of such vortices was predicted theoretically by Görtler (1940) while Gregory & Walker (1950) demonstrated experimentally the occurrence of this form of instability on the flap of a Griffith suction aerofoil. The instability arises because of centrifugal effects and has much in common with the Taylor-vortex instability of fully developed circumferential flows between concentric cylinders. It is clear, however, that some difference between the problems must exist because of the developing nature of the basic flow in which Görtler vortices occur. Indeed, the non-parallel nature of the basic flow in the Görtler problem causes the greatest difficulty in formulating a mathematically tractable and justifiable linear-stability eigenvalue problem. We shall see in this paper that for vortices having wavelength of the same (or smaller) order of magnitude as the boundary-layer thickness there is no justification for making the parallel-flow approximation.

Görtler (1940) investigated the stability of boundary-layer flows on concave walls by making the parallel-flow approximation and neglecting higher-order curvature effects. Thus Görtler derived a sixth-order ordinary differential system essentially identical to the system appropriate to the Taylor-vortex stability equations discussed by Taylor (1923). These equations were solved approximately by Green-function methods, and Görtler found that boundary layers on curved walls are unstable at sufficiently high flow speeds. Moreover, Görtler found a critical value for the non-dimensional parameter G (essentially a Taylor number) subsequently referred to as the Görtler number and above which growing disturbances to the basic state exist. This critical value of G was found by Görtler to occur at a finite value of the wavenumber of the vortices.

Hämmerlin (1955) solved Görtler's equations more rigorously and found the surprising result that the critical Görtler number corresponded to vortices of infinite wavelength. In an attempt to rectify this situation, Hämmerlin (1956) derived and solved a modified version of Görtler's equations that included some higher-order curvature terms. Hämmerlin found that this alternative set of equations had the required property that the most dangerous disturbance occurred at a wavenumber $k \neq 0$. An alternative set of 'improved' equations was derived by Smith (1955), who included not only the higher-order curvature terms retained by Hämmerlin (1956) but also certain terms arising from the non-parallel nature of the basic flow. The results of Smith's calculations were essentially identical with those of Hämmerlin (1956). Since that time, numerous authors have investigated various modified versions of the stability equations of Hämmerlin and Smith. A review of the results of some of these investigations can be found in Herbert (1976) and Floryan & Saric (1979). In the latter paper, in addition to a comprehensive review of previous work, the authors determine the effect of suction on Görtler-vortex instability. It appears that at small values of the non-dimensional wavenumbers of the vortices there is no limit to the disagreement between the different calculations.

We shall show in this paper that this result is due to the effect of boundary-layer growth. Moreover, we shall show that the parallel-flow approximation universally applied in previous investigations of Görtler vortices has no mathematical justification except at high values of k. We shall therefore develop a formal asymptotic expansion of the appropriate linear-stability partial differential equations based on the smallness of the wavelength of the imposed disturbance. The expansion method that we use has many similarities with the procedure used by Smith (1979, 1980), who investigated the effect of boundary-layer growth on Tollmien–Schlichting waves in developing flows.

The surprising outcome of our calculation is that the disturbances at high wavenumbers are concentrated in internal viscous layers away from the wall and the free stream. The vertical structure of the disturbances in these layers can be expressed simply in terms of parabolic cylinder functions, and a simple analytic expression for the neutral curve at high wavenumbers is obtained. Moreover, non-parallel effects can be taken care of in a rational way using our expansion procedure.

In order to see how the asymptotic structure of the high-wavenumber disturbance equations depends on the precise nature of the basic flow we shall first of all return to the classical Taylor-vortex problem and the related problem discussed by Dean (1928). The latter author investigated the stability of pressure gradient driven flows between concentric cylinders and found that such flows could also be unstable to Taylor vortices. Thus in §2 we shall investigate asymptotic approximations to these problems for the case when the vortices have wavelengths small compared with the separation of the cylinders. We shall see that the nature of the disturbances in this limit depends crucially on the basic circumferential flow $\overline{V}(r)$. It is found that the disturbances are concentrated in thin viscous layers centred on the position where $|\overline{V}\overline{V}'|$ has its maximum value. Thus in the Taylor problem the viscous layers are at the inner cylinder, while in the Dean problem they are in the interior of the fluid. In fact the structure of the disturbance velocity field in the Dean problem suggests the form of the corresponding velocity field in the boundary layer case.

In §3 we shall use the asymptotic procedure formulated for the Dean problem in

§2 to investigate high-wavenumber Görtler vortices in growing boundary layers. In §4 we discuss the results obtained in §3 and their relevance to the available experimental results.

2. Fully developed flows

In this section we investigate the short-wavelength asymptotic structure of the centrifugal instabilities of two fundamental flows between concentric cylinders of radii R and $R+d$. The first problem has a basic circumferential flow driven by the steady motion of the inner cylinder about its axis, while the outer cylinder is held fixed. In the second problem, first considered by Dean (1928), the basic flow is driven by a constant pressure gradient in the circumferential direction. We shall refer to the former and latter problems as the Taylor and Dean problems respectively.

For convenience we restrict our attention to the case when $d/R \ll 1$ but, apart from simplifying the analysis, there is no great difference between this limit and the finite-gap case. The basic flows set up in the two problems are susceptible to Taylor vortex instabilities for which the perturbation velocities are all proportional to $\exp i\{z/\epsilon\}$, with ϵ^{-1} a non-dimensional wavenumber, while z is the axial variable scaled on d. The appropriate linear-stability equations are well known (see e.g. Dean 1928; Chandrasekhar 1958; Stuart 1963) and can be written in the form

$$\left[\epsilon^2\frac{d^2}{dy^2}-1\right]^2 u = \epsilon^2 T\bar{V}v, \tag{2.1a}$$

$$\left[\epsilon^2\frac{d^2}{dy^2}-1\right] v = \epsilon^2 u\frac{d\bar{V}}{dy}, \tag{2.1b}$$

$$\epsilon\frac{du}{dy}+iw = 0, \tag{2.1c}$$

together with the boundary conditions

$$u = v = w = 0, \quad y = 0, 1. \tag{2.2}$$

Here u, v and w are suitably dimensionless velocities in the radial, azimuthal and axial directions, while the radial variable y is scaled on d such that the inner and outer cylinders correspond to $y=0$ and $y=1$ respectively. We are concerned only with neutrally stable perturbations, so that there is no growth rate appearing in (2.1). The parameter T is the Taylor number defined by

$$T = \frac{2V_m^2 d^3}{R\nu^2}, \tag{2.3}$$

where V_m is the speed of the inner cylinder in the Taylor problem and the mean basic circumferential fluid velocity in the Dean problem. The function $\bar{V}(y)$ appearing in (2.1) is the basic circumferential velocity field, and is given by

$$\bar{V} = \bar{V}_1(y) = 1-y \tag{2.4}$$

in the Taylor problem and

$$\bar{V} = \bar{V}_2(y) = 6y(1-y) \tag{2.5}$$

in the Dean problem.

The eigenrelation $\epsilon = \epsilon(T)$ specified by (2.1) and (2.2) can be solved numerically to determine the neutral curve in the (ϵ, T)-plane. In figure 1 we have shown the neutral curve for Dean's problem, and we see that for $T > T_{c1} = 2581$ there is a finite

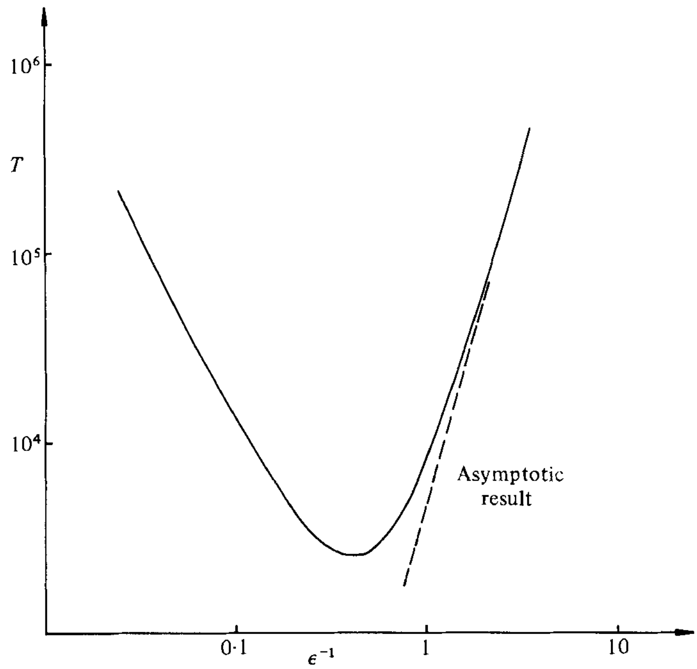

FIGURE 1. A comparison between the neutral curve for the Dean problem and its asymptotic approximation at high wavenumbers.

bandwidth of unstable disturbances. In fact it is believed that there exists an infinite sequence of neutral curves, the nth neutral curve having a minimum value of $T = T_{cn}$ greater than T_{c1}, and the neutral curves are ordered such that $T_{c1} > T_{c2} > T_{c3} \ldots$. The eigenrelation for the Taylor problem has similar properties, but clearly the values of T_{c1}, T_{c2} etc. in this case are different.

Our concern here is with the asymptotic structure of the right-hand branches of these neutral curves and the corresponding eigenfunctions. This structure will be used in §3, where we shall then be able to determine the effect of boundary-layer growth on Görtler vortices. The right-hand branches of the neutral curves correspond to small values of ϵ, so, following Meksyn (1946), we consider the limit $\epsilon \to 0$, in which case it is natural to seek a WKB solution of (2.1) with

$$u = E \sum_0^\infty \epsilon^n u_n(y), \tag{2.6a}$$

$$v = E \sum_0^\infty \epsilon^n v_n(y), \tag{2.6b}$$

$$w = E \sum_0^\infty \epsilon^n w_n(y), \tag{2.6c}$$

where
$$E = \exp \frac{i}{\epsilon} \int_{y_0}^{y} K(\theta)\, d\theta.$$

Here y_0 is some constant in the interval $[0, 1]$ while $K(\theta)$, u_0, u_1 etc. are to be determined. For small values of ϵ the dominant terms on the right-and left-hand sides of (2.1b) balance if $T \sim \epsilon^{-4}$ and $U/V \sim O(\epsilon^{-2})$. We therefore expand T in the form

$$T = \sum_0^\infty \lambda_n \epsilon^{-4+n}; \tag{2.6d}$$

then $K(\theta)$ is determined by equating terms of order ϵ^2 in (2.1), after first substituting the above expansions into (2.1). We obtain the equation

$$\{K^2+1\}^3 = -\lambda_0 \bar{V}\bar{V}', \tag{2.7}$$

which gives six possible values for K. However an examination of the structure of the eigenfunctions corresponding to the six roots of (2.7) shows that only real values of K lead to acceptable solutions of the stability equations. Such a result can also be inferred from the results of Meksyn (1946).

The condition that (2.7) should have some real roots imposes a condition on the velocity field $\bar{V}(y)$ and a further condition on λ_0. We can see directly from (2.7) that real values of K can only occur in regions where $\bar{V}\bar{V}'$ is negative and

$$\lambda_0 > -\{\bar{V}\bar{V}'\}^{-1}_{\text{min}} \tag{2.8}$$

where $\{\bar{V}\bar{V}'\}_{\text{min}}$ denotes the minimum value of $\bar{V}\bar{V}'$. In the Taylor problem $\bar{V}\bar{V}'$ is negative, except at $y = 1$ where $\bar{V}$ is zero, while in the Dean problem $\bar{V}\bar{V}'$ is negative only in $(\frac{1}{2}, 1)$. Moreover, in the former problem real values of K exist only in the interval $[0, y_0^*]$ for any λ_0 satisfying the above inequality and where $y_0^* = 1-\lambda_0^{-1}$. In the Dean problem the corresponding interval is $[y_1^*, y_2^*]$, where y_1^*, y_2^* are roots of the equation

$$36y(1-y)(1-2y) = -\lambda_0^{-1} \tag{2.9}$$

in the interval $(\frac{1}{2}, 1)$. Thus a difference in the structure of the WKB solutions for the two problems is already apparent and could have been anticipated on the basis of Rayleigh's criterion for inviscid flows. We shall return to this point later.

The WKB solutions fail when the real roots of (2.7) coalesce, so turning points will exist near y_0^*, y_1^* and y_2^*. These turning points are of the first order and the structure of the disturbances in $\epsilon^{\frac{2}{3}}$ neighbourhoods of each turning point can be found in the usual way in terms of Airy functions. All the eigenvalues resulting from the WKB solutions necessarily have $\lambda_0 > -\{\bar{V}\bar{V}'\}^{-1}_{\text{min}}$ and it is found that an infinite number of eigenvalues λ_0 exists in each problem. We note that if $\lambda_0 = -\{\bar{V}\bar{V}'\}^{-1}_{\text{min}}$ the turning point in the Taylor problem has moved to $y = 0$, while in the Dean problem $y_1^* = y_2^* = \frac{1}{2}\{1+\sqrt{\frac{1}{3}}\}$. In both cases the WKB method breaks down, since there is now no finite interval where (2.7) has real solutions. We shall see below that when $\lambda_0 = -\{\bar{V}\bar{V}'\}^{-1}_{\text{min}}$ solutions of the original disturbance equations can still be constructed and are the most dangerous modes available.

We note, however, that in the Dean problem when $y_1^* = y_2^*$ the appropriate transition interval corresponds to a second-order turning point and so the interval is of thickness $\epsilon^{\frac{1}{2}}$. Thus in this interval the velocity field will be expressed in terms of parabolic-cylinder functions, while in the Taylor case the turning point at $y = 0$ remains of the first order so that Airy-function solutions occur. Thus the structures of the most dangerous modes in the two problems are quite distinct and result from the manner in which Rayleigh's criterion is violated in each problem.

We recall that for a circumferential velocity field $(0, V(r), 0)$ the flow is inviscidly unstable if V^2/r increases towards the centre of curvature of the streamlines. In the present small-gap case it follows that locally the velocity field $\bar{V}(y)$ violates Rayleigh's criterion the most where $|\bar{V}\bar{V}'|$ has its greatest value. In the Taylor problem this occurs at $y = 0$, while in the Dean problem it occurs at the interior point where $y_1^* = y_2^* = \frac{1}{2}(1+\sqrt{\frac{1}{3}})$. Thus we expect that short-wavelength Taylor vortices will be localized near $y = 0$ in the Taylor problem and near $y = \frac{1}{2}(1+\sqrt{\frac{1}{3}})$ in the Dean problem. It remains now for us to work out the details for each of these problems for the case $\lambda_0 = -\{\bar{V}\bar{V}'\}_{\text{min}}$; the WKB solutions will be omitted for the sake of brevity but are available on request from the author.

We first consider the Taylor problem, in which case it is convenient to eliminate u from (2.1) and (2.2) to give

$$\left[\epsilon^2 \frac{d^2}{dy^2} - 1\right]^3 v = -\epsilon^4 T(1-y)\, v, \tag{2.10a}$$

$$v = \frac{d^2 v}{dy^2} = \frac{d}{dy}\left\{\epsilon^2 \frac{d^2}{dy^2} - 1\right\} v = 0, \quad y = 0, 1. \tag{2.10b}$$

We now expand the Taylor number in the form

$$T = \epsilon^{-4}\{1 + \gamma_1 \epsilon^{\frac{2}{3}} + \ldots\},$$

and define the variable ξ by

$$\xi = \frac{y}{3^{\frac{1}{3}}\epsilon^{\frac{2}{3}}}.$$

We then write

$$v = v_0(\xi) + \epsilon^{\frac{2}{3}} v_1(\xi) + \ldots \tag{2.11}$$

and substitute this expression into (2.10a). If we then equate terms of order $\epsilon^{\frac{2}{3}}$ after replacing y by $\xi \epsilon^{\frac{2}{3}} 3^{\frac{1}{3}}$ we obtain

$$\frac{d^2 v_0}{d\xi^2} - \xi v_0 + \gamma_1 3^{-\frac{1}{3}} v_0 = 0.$$

The solution of this equation that is bounded as $\xi \to \infty$ is

$$v_0 = C \operatorname{Ai}(\xi - \gamma_1 3^{-\frac{1}{3}}), \tag{2.12}$$

where C is an arbitrary constant. This solution does not satisfy the required boundary conditions at $y = 0$. Thus it is necessary to look for another boundary layer embedded inside the $\epsilon^{\frac{2}{3}}$ layer. The appropriate layer is of thickness ϵ, so we define the variable Y by

$$Y = y\epsilon^{-1},$$

and in this layer we expand v in the form

$$v = \epsilon^{\frac{1}{3}} V_0(Y) + \epsilon^{\frac{2}{3}} V_1(Y) + \ldots.$$

We now substitute this expression into (2.10) and equate terms of order $\epsilon^{\frac{1}{3}}$ to give

$$\left\{\frac{d^2}{dY^2} - 1\right\}^3 V_0 = -V_0, \tag{2.13a}$$

$$V_0 = \frac{d^2 V_0}{dY^2} = \frac{d^3 V_0}{dY^3} - \frac{dV_0}{dY} = 0 \quad (Y = 0). \tag{2.13b}$$

The solution of (2.13) that does not grow exponentially with Y is

$$V_0 = a_0 + a_1 Y + \exp\left[-(3^{\frac{1}{4}} \cos \tfrac{1}{12}\pi) Y\right]\{a_2 \sin (3^{\frac{1}{4}} \sin \tfrac{1}{12}\pi) Y + a_3 \cos (3^{\frac{1}{4}} \sin \tfrac{1}{12}\pi) Y\}.$$

We can choose a_1, a_2 and a_3 in terms of a_0 such that the required conditions at $Y = 0$ are satisfied. For large values of Y we see that $V_0 \sim Y$, so that matching with (2.11) can only be achieved if

$$\operatorname{Ai}(-\gamma_1 3^{-\frac{1}{3}}) = 0, \tag{2.14}$$

which gives an infinite sequence of eigenvalues $\{\gamma_{1n}\}$ corresponding to the zeros of the Airy function on the negative real axis. The most dangerous mode of instability has T given by

$$T = \epsilon^{-4}\{1 + 3{\cdot}372\epsilon^{\frac{2}{3}} + \ldots\}. \tag{2.15}$$

For large values of m the eigenvalues can be approximated by

$$2(\tfrac{1}{3}\gamma_{1m})^{\frac{3}{2}} \sim \pi\{m-\tfrac{1}{4}\}, \tag{2.16a}$$

and the Taylor-number expansion then takes the form

$$T = \epsilon^{-4}\{1+\epsilon[(\tfrac{3}{2})^{\frac{3}{2}}\pi(m-\tfrac{1}{4})]^{\frac{2}{3}}+\dots\}. \tag{2.16b}$$

For such modes the disturbance is strongest for $\xi \sim m^{\frac{2}{3}}$, so the higher modes become concentrated progressively further away from the inner cylinder. In between the wall and the region $\xi \sim m^{\frac{2}{3}}$ the disturbance is almost periodic and decaying slowly in amplitude towards the wall. In fact, when $m \sim \epsilon^{-1}$ the analysis breaks down and the WKB expansion must be used.

We now return to the Dean problem and find the asymptotic form for the most dangerous linear disturbances. We recall that the WKB formulation for this problem fails when y is equal to either of the roots of (2.9) lying in $(\frac{1}{2}, 1)$. These roots coalesce at $y = \frac{1}{2}\{1+\sqrt{\frac{1}{3}}\}$ when $\lambda_0 \to 1+(2\sqrt{3})^{-1}$, and in the neighbourhood of this point the disturbance velocity field is expressed in terms of parabolic-cylinder functions. We first write

$$T = \epsilon^{-4}\{\mu_0+\mu_1\epsilon^{\frac{1}{2}}+\dots\}, \tag{2.17}$$

and define the variable η by

$$\eta = \epsilon^{-\frac{1}{2}}\{y-\tfrac{1}{2}[1+\sqrt{\tfrac{1}{3}}]\}. \tag{2.18}$$

We now replace $\overline{V}$ in (2.1 a,b) by $6y(1-y)$ and then write the two differential equations in terms of the variable η. The functions u and v are then written as

$$u = \{u_0(\eta)+\epsilon^{\frac{1}{2}}u_1(\eta)+\dots\}, \tag{2.19a}$$

$$v = \epsilon^2\{v_0(\eta)+\epsilon^{\frac{1}{2}}v_1(\eta)+\dots\}, \tag{2.19b}$$

where we have anticipated that $u/v \sim O(\epsilon^{-2})$ in this region. At order ϵ^0 we find that u_0 and v_0 must satisfy the equations

$$u_0 = \mu_0 v_0, \quad v_0 = 2\sqrt{3}\,u_0,$$

and so for a consistent solution of these equations we must choose $\mu_0 = (2\sqrt{3})^{-1}$. At order $\epsilon^{\frac{1}{2}}$ we obtain

$$u_1 = \mu_0 v_1+\mu_1 v_0-2\sqrt{3}\,\eta\mu_0 v_0,$$
$$v_1 = 2\sqrt{3}\,u_1+12\eta u_0,$$

and for a consistent solution of this system we must choose $\mu_1 = 0$. At order ϵ we find

$$u_2 = \mu_0 v_2-2\sqrt{3}\,\eta\mu_0 v_1+2\frac{d^2u_0}{d\eta^2}+\mu_2 v_0-6\eta^2\mu_0 v_0,$$

$$v_2 = 2\sqrt{3}\,u_2+12\eta u_1+\frac{d^2v_0}{d\eta^2}.$$

These equations can be simplified by replacing u_0 by $\mu_0 v_0$ and u_1 by $\{\mu_0 v_1-2\sqrt{3}\,\eta\mu_0 v_0\}$. The resulting linear equations for u_2 and v_2 have a consistent solution if

$$\frac{d^2v_0}{d\eta^2}+\frac{2}{\sqrt{3}}\mu_2 v_0-6\eta^2 v_0 = 0, \tag{2.20}$$

which is the parabolic-cylinder equation. This equation has a set of solutions that decay as $|\eta| \to \infty$ if μ_2 is given,

$$\mu_2 = \mu_{2n} = 2^{\frac{1}{2}}3[\tfrac{1}{2}+n] \quad (n = 0,1,2,\dots), \tag{2.21}$$

and then V_0 is given by

$$V_0 = V_{0n} = e^{-\sqrt{\frac{3}{2}}\eta^2}\,\mathrm{He}_n((24)^{\frac{1}{4}}\eta)$$

where He_n is a Hermite polynomial. It follows from (2.17), (2.21) that the most dangerous linear mode has a Taylor-number expansion

$$T = \frac{\epsilon^{-4}}{2\sqrt{3}}\{1+3\sqrt{6}\,\epsilon+\ldots\}, \tag{2.22}$$

for small values of ϵ. For large values of n the disturbance velocity field spreads out and becomes strongest near $\eta = \pm n^{\frac{1}{2}}$ and is almost periodic between these points. If n is taken to be of order ϵ^{-1} the present formulation breaks down and the WKB method must be used.

The asymptotic structure derived above for the Dean problem will in §3 form the basis for an asymptotic description of short-wavelength disturbances in boundary layers. Thus, following the suggestion of an anonymous referee, we have in figure 1 shown the two-term asymptotic approximation to the neutral curve. There seems little doubt that our asymptotics adequately describe the most dangerous form of instability at small wavelengths. We cannot, however, use this curve to assess the range of validity for a similar asymptotic description of the Görtler problem. Indeed it is not clear that a unique neutral curve for all wavenumbers should even exist in the Görtler problem, since the neutral curves given by previous investigators result from approximations that cannot be justified. The results shown in figure 1 would suggest that, if a unique neutral curve does exist for the Görtler problem, then at $O(1)$ wavenumbers the asymptotics would probably only give qualitative information about the instability. The referee also made available to the author the results of a numerical investigation of the Taylor problem at large wavenumbers. The numerical results supplied suggest that $\lim_{\epsilon\to 0} T\epsilon^4 \approx 1{\cdot}04$, thus giving a check on (2.15).

Finally we close this section with a brief discussion of how the expansions (2.15) and (2.22) compare with the corresponding expansions of the Rayleigh number R in terms of the wavenumber a for the related Bénard convection problem. In the latter problem the differential equation to be solved is simply (2.10a) with $1-y$ replaced by 1, which gives a differential equation with constant coefficients. Thus the basic flow is equally unstable everywhere in (0, 1) and so the disturbances for large a are not concentrated in boundary layers. If free-surface conditions are applied at $y = 0, 1$ then the eigenrelation of the most dangerous mode can be written in the form

$$R = a^4\left\{1+\frac{3\pi^2}{a^2}+\frac{3\pi^4}{a^4}+\frac{\pi^6}{a^6}\right\}, \tag{2.23}$$

which is exact and does not have the same form as (2.15) or (2.22). If rigid boundary conditions are used then the velocity field has boundary layers of thickness a^{-1} at $y = 0, 1$. In between these layers the disturbance is periodic and the expansion of the Rayleigh number takes the form

$$R = a^4 f_0(a) + O(a^3), \tag{2.24}$$

where $f_0(a)$ is of order a^0 and satisfies a transcendental equation. Thus, as pointed out to the author by the referee, a unifying feature of the Bénard, Taylor and Dean problems (also, as we shall see in §3, the Görtler problem) is that at high wavenumbers the stability parameter in the neutral case is proportional to the fourth power of the wavenumber. The higher-order dependence on the wavenumber varies between the problems, as does the zeroth-order structure of the disturbance.

3. Görtler vortices in a growing boundary layer

We consider a fluid flow over a section of a concave wall having constant radius of curvature R. If we now take l to be a typical length measured along the curved wall, then we define

$$\delta = lR^{-1}, \tag{3.1}$$

and we assume that $\delta \ll 1.$

We choose dimensionless variables s and ζ such that

$$s = \theta'\delta^{-1},$$
$$\zeta = -\{r'-R\}\,l^{-1},$$

where (r', θ, z') are cylindrical polar coordinates with $r' = 0$ corresponding to the axis of the cylindrical surface. We now suppose that a high-Reynolds-number flow is set up over this surface. For definiteness we assume that a second cylindrical surface is present at $r = R-l$ and that at $\theta = 0$ a flow with constant velocity V_0^* in the azimuthal direction enters the channel. If the Reynolds number Re, defined by

$$Re = \frac{V_0^* l}{\nu}$$

is large, then boundary layers scaled on $Re^{-\frac{1}{2}}$ will be set up at each wall, together with an inviscid core flow. We focus our attention only on the boundary layer on the outer wall, and define a variable y by

$$y = Re^{\frac{1}{2}}\zeta;$$

then for $Re \gg 1$ and $\delta \ll 1$ the basic flow in this boundary layer has velocities u and v in the y- and s-directions given by

$$\frac{v}{V_0^*} = v_{\mathrm{B}} = \overline{V}_0(\xi) + O(Re^{-\frac{1}{2}}, \delta), \tag{3.2a}$$

$$\frac{u}{V_0^*} = Re^{-\frac{1}{2}} u_{\mathrm{B}} = Re^{-\frac{1}{2}}\{\overline{V}_0(\xi_1 s) + O(Re^{-\frac{1}{2}}, \delta)\}, \tag{3.2b}$$

where $\xi = y(2s)^{-\frac{1}{2}}$, and $\overline{U}_0$, $\overline{V}_0$ satisfy

$$\overline{V}_0 = f'(\xi), \quad \overline{U}_0 = \{f(\xi) - \xi f'(\xi)\}\,(2s)^{-\frac{1}{2}}.$$

Here $f(\xi)$ satisfies the usual ordinary differential system

$$f''' + ff'' = 0,$$
$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

The inviscid core and the other boundary layer play no role in the analysis that we give here and so we do not describe them further. However, it follows that the analysis which we give applies directly to any flow over a concave surface that leads to a velocity field of the form (3.2).

We now perturb the flow in the boundary layer near $\zeta = 0$ such that the disturbed flow is

$$\frac{u}{V_0^*} = Re^{-\frac{1}{2}}\{u_{\mathrm{B}} + \gamma U(y, s, z, t)\}, \tag{3.3a}$$

$$\frac{v}{V_0^*} = v_{\mathrm{B}} + \gamma V(y, s, z, t), \tag{3.3b}$$

$$\frac{w}{V_0^*} = Re^{-\frac{1}{2}} \gamma W(y, s, z, t), \tag{3.3c}$$

where z is a dimensionless axial variable scaled on the boundary-layer thickness such that
$$z = z'l^{-1}Re^{\frac{1}{2}},$$
and t is a non-dimensional time variable scaled on lV_0^{*-1}. Suppose next that the Navier–Stokes equations written in cylindrical polar coordinates are rewritten in terms of the dimensionless variables y, s and z, and u, v, w are replaced by (3.3). If we equate terms of order γ in the resulting equations we obtain

$$\frac{-\partial U}{\partial t}+U_{yy}+U_{zz}+P_y+G\bar{V}_0 V = [-(\bar{U}_0 U)_y+\bar{V}_0 U_s+V\bar{U}_{0s}]+O(Re^{-\frac{1}{2}},\delta), \quad (3.4a)$$

$$\frac{-\partial V}{\partial t}+V_{yy}+V_{zz}+U\bar{V}_{0y} = [-\bar{U}_0 V_y+\bar{V}_0 V_s+V\bar{V}_{0s}]+O(Re^{-\frac{1}{2}},\delta), \quad (3.4b)$$

$$\frac{-\partial W}{\partial t}+W_{yy}+W_{zz}-P_z = [-\bar{U}_0 W_y+\bar{V}_0 W_s]+O(Re^{-\frac{1}{2}},\delta), \quad (3.4c)$$

$$-U_y+W_z = -[V_s]+O(\delta), \quad (3.4d)$$

where P, the pressure perturbation corresponding to (U, V, W), is scaled on Re^{-1}. The parameter G is the Görtler number given by
$$G = 2Re^{\frac{1}{2}}\delta, \quad (3.5)$$
which is first held fixed while $Re^{-\frac{1}{2}} \to 0$ and $\delta \to 0$. We shall shortly consider the second limit $G \to \infty$. The range of validity of our expansions will be discussed later. The terms in the square brackets above represent the effect of boundary-layer growth and we see that the presence of the terms involving s-derivatives of U_0, U etc. causes the linear-stability problem for a disturbance periodic in z to be a partial differential system in the three variables t, y and s. Thus there is no mathematical justification for ignoring these terms or, alternatively, as is often the case, ignoring only those involving s-derivatives of U, V and W.

However, we now show that by taking the further limit $G \to \infty$ we can obtain a formal asymptotic expansion to this system. The essential idea is that when $G \to \infty$ the terms in the square brackets only enter the hierarchy of resulting differential systems at higher order. This enables us to solve these systems by Fourier expanding in z, so that only ordinary differential systems are encountered. We stress that such a procedure can only be carried out if the terms of order $Re^{-\frac{1}{2}}, \delta$ in (3.4) remain negligible compared with those which are retained. This is in fact the case and we return to this point later.

Suppose next that U, V, W and P are all taken to be proportional to $e^{iz/\epsilon}$, where ϵ^{-1}, the non-dimensional wavenumber of the disturbance, is large. In the absence of the terms on the right-hand sides of (3.4) these equations are similar to those appropriate to fully developed flows in curved channels. Our experience in §2 shows that the required asymptotic expansions depend crucially on the position where $\bar{V}_0 \bar{V}_{0y}$ has its maximum value. For the Blasius boundary layer this quantity has a maximum at
$$\xi = \xi^+ = 1{\cdot}53, \quad (3.6)$$
so we must base our expansions on those used in §2 for the Dean problem. Thus we focus our attention on the region of thickness $\epsilon^{\frac{1}{2}}$ centred on $\xi = \xi^+$ in which $O(U) = O(\epsilon^{-2}V)$. It follows from the equation of continuity that $O(W) = O(\epsilon^{+\frac{1}{2}}U)$ and from the z-momentum equation that $O(P) = O(\epsilon^{-\frac{1}{2}}U)$. We now define the variable η by
$$\eta = \epsilon^{-\frac{1}{2}}\{\xi-\xi^+\}, \quad (3.7a)$$

so that in (3.4) we must replace $\partial/\partial y$ by $(2s\epsilon)^{-\frac{1}{2}}\partial/\partial\eta$ and $\partial/\partial s$ by

$$\frac{\partial}{\partial s}\frac{-[\xi^{+}+\epsilon^{\frac{1}{2}}\eta]}{2s\epsilon^{\frac{1}{2}}}\frac{\partial}{\partial \eta}.$$

In view of the fact that the boundary layer is growing with s, there is some ambiguity about how we should define neutral stability (for a discussion of this point in connection with the Tollmien–Schlichting mode of instability for the Blasius boundary layer see, e.g. Gaster 1974; Smith 1979). Since the disturbance has a lengthscale $O(\epsilon)$ in the z-direction we can expect that the spatial growth rate of a Gortler vortex is on a lengthscale such that $\partial^2/\partial z^2 \sim \partial/\partial s \sim 1/\epsilon^2$ in (3.4). Thus we assume that U, V, W, P are all proportional to E, where

$$E = \exp\left\{\frac{iz}{\epsilon}+\frac{1}{\epsilon^2}\int^{s}[\beta_0(\phi)+\epsilon^{\frac{1}{2}}\beta_1(\phi)+\ldots]\,d\phi\right\}, \tag{3.7b}$$

and the growth rate $(\beta_0+\epsilon^{\frac{1}{2}}\beta_1+\ldots)$ is to be evaluated. We may assume that in the initial stages of the instability β_0, β_1 etc. are all real and U, V, W and P are independent of time. The ambiguity in the condition for neutral stability arises because U, V, W and P have different dependences on the slow variable s. Thus if we expand U, V, W and P in the form

$$U = \{U_0(\eta,s)+\epsilon^{\frac{1}{2}}U_1(\eta,s)+\ldots\}E+\text{C.C.}, \tag{3.8a}$$

$$V = \{\epsilon^{2}V_0(\eta,s)+\epsilon^{\frac{5}{2}}V_1(\eta,s)+\ldots\}E+\text{C.C.}, \tag{3.8b}$$

$$W = \{\epsilon^{\frac{1}{2}}W_0(\eta,s)+\epsilon W_1(\eta,s)+\ldots\}E+\text{C.C.}, \tag{3.8c}$$

$$P = \{\epsilon^{-\frac{1}{2}}P_0(\eta,s)+P_1(\eta,s)+\ldots\}E+\text{C.C.}, \tag{3.8d}$$

replace G by the expansion

$$G = \epsilon^{-4}\{g_0+\epsilon^{\frac{1}{2}}g_1+\epsilon g_2+\ldots\}, \tag{3.9}$$

and take V evaluated at $\eta=\eta^*$, $z=z^*$ to be a representative disturbance quantity, then the relative change of V with respect to s is

$$\frac{\partial V/\partial s}{V} = \frac{\beta_0}{\epsilon^2}+\frac{\beta_1}{\epsilon^{\frac{3}{2}}}+\frac{\beta_2}{\epsilon}+\frac{1}{\epsilon^{\frac{1}{2}}}\left[\beta_3\frac{-\partial V_0/\partial\eta(\eta^*,s)\,\xi^{+}}{2sV_0(\eta^*,s)}\right]+\ldots.$$

Thus for neutral stability we must choose the first four β_j such that

$$\beta_0=\beta_1=\beta_2=0,\quad \beta_3=\frac{\partial V_0}{\partial\eta}(\eta^*,s)\,\xi^{+}\Big/2sV_0(\eta^*,s), \tag{3.10}$$

and the last condition clearly depends on η^* and would, of course, be different if we, say, choose W to be the representative disturbance quantity. However, since our principal aim is to discuss the neutral case we will set $\beta_0=\beta_1=\beta_2=0$ in the following analyses. This considerably reduces the amount of algebra to be carried out. The coefficients $g_0,\ldots$ appearing in (3.9) are functions of s to be determined such that the flow is locally neutral at the location s. Alternatively we could take g_0, g_1 etc. to be given and then find $s=s(\epsilon,G)$ such that the flow is neutral at this location.

We now expand $\bar{V}_0$ and $\bar{U}_0$ in the neighbourhood of ξ^{+} by writing

$$\bar{V}_0 = V_{00}^{+}+\epsilon^{\frac{1}{2}}\eta V_{01}^{+}+\epsilon\eta^2 V_{02}^{+}+\ldots, \tag{3.11a}$$

$$\bar{U}_0 = \{2s\}^{-\frac{1}{2}}\{U_{00}^{+}+\epsilon^{\frac{1}{2}}U_{01}^{+}\eta+\epsilon\eta^2U_{02}^{+}+\ldots\}, \tag{3.11b}$$

where

$$V_{0j}^{+} = (j!)^{-1}\frac{\partial^j\bar{V}_0}{\partial\xi^j}(\xi^{+}),\quad U_{0j}^{+} = (j!)^{-1}\frac{\partial^j\bar{U}_0}{\partial\xi^j}(\xi^{+}). \tag{3.11c}$$

We now substitute the expansions (3.8), (3.9), (3.10) into (3.4) and equate like powers of $\epsilon^{\frac{1}{2}}$ after replacing $\partial/\partial y$ by $(2s\epsilon)^{-\frac{1}{2}}\partial/\partial\eta$ and $\partial/\partial s$ by

$$\frac{\partial}{\partial s}\frac{-[\xi^+ + \epsilon^{\frac{1}{2}}\eta]}{2s\epsilon^{\frac{1}{2}}}\frac{\partial}{\partial\eta}.$$

If we restrict our attention to the first four terms in the expansion of (3.4) we obtain

$$\mathbf{A}\mathbf{U}_0 = 0, \tag{3.12a}$$

$$\mathbf{A}\mathbf{U}_1 = \begin{bmatrix} V_0\{g_1 V_{00}^+ + \eta g_0 V_{01}^+\} \\ 2\eta V_{02}^+ U_0/(2s)^{\frac{1}{2}} \\ 0 \\ 0 \end{bmatrix}, \tag{3.12b}$$

$$\mathbf{A}\mathbf{U}_2 = \begin{bmatrix} \frac{1}{2s}U_{0\eta\eta} + \frac{1}{(2s)^{\frac{1}{2}}}P_{0\eta} + V_1\{g_1 V_{00}^+ + \eta g_0 V_{01}^+\} + V_0\{\eta^2 g_0 V_{02}^+ + \eta g_1 V_{01}^+ + g_2 V_{00}^+\} \\ 2\eta V_{02}^+ U_1/(2s)^{\frac{1}{2}} + 3\eta^2 U_0 V_{03}^+/(2s)^{\frac{1}{2}} + \frac{1}{2}s V_{0\eta\eta} \\ W_{0\eta\eta}/2s \\ 0 \end{bmatrix} \tag{3.12c}$$

$$\mathbf{A}\mathbf{U}_3 = \begin{bmatrix} \begin{aligned} &\Big\{\frac{1}{2s}U_{1\eta\eta} + \frac{1}{(2s)^{\frac{1}{2}}}P_{1\eta} + V_2(g_1 V_{00}^+ + g_0\eta V_{01}^+) + V_1(g_0\eta^2 V_{02}^+ + g_1\eta V_{01}^+ + g_2 V_{00}^+) \\ &\quad -\beta_3 U_0 V_{00}^+ + V_0(g_3 V_{00}^+ + g_2 V_{01}^+\eta + g_1 V_{02}^+\eta^2 + \eta^3 g_0 V_{03}^+) \\ &\qquad + \frac{U_{00}^+}{2s}U_{0\eta} + \frac{V_{00}^+}{2s}\xi^+ U_{0\eta}\Big\} \end{aligned} \\ \begin{aligned} &\Big\{\frac{2\eta}{(2s)^{\frac{1}{2}}}U_2 V_{02}^+ - \beta_3 V_0 V_{00}^+ + V_{1\eta\eta}/2s + \frac{3\eta^2}{(2s)^{\frac{1}{2}}}U_1 V_{03}^+ \\ &\qquad + \frac{4\eta^3}{(2s)^{\frac{1}{2}}}U_0 V_{04}^+ + \frac{U_{00}^+}{2s}V_{0\eta} + \frac{V_{00}^+\xi^+}{2s}V_{0\eta}\Big\} \end{aligned} \\ \{W_{1\eta\eta}/2s + U_{00}^+ W_{0\eta}/2s + V_{00}^+\xi^+ W_{0\eta}/2s - V_{00}^+\beta_3 W_0\} \\ 0 \end{bmatrix} \tag{3.12d}$$

where $\mathbf{U}_j = (U_j, V_j, W_j, P_j)^{\mathrm{T}}$, and the matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{bmatrix} 1 & -g_0 V_{00}^+ & 0 & 0 \\ \frac{-V_{01}^+}{(2s)^{\frac{1}{2}}} & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ \frac{-\partial/\partial\eta}{(2s)^{\frac{1}{2}}} & 0 & i & 0 \end{bmatrix}. \tag{3.13}$$

It remains now for us to solve (3.12a–d) successively in order to determine the coefficients in the expansion of the Görtler number. In each of these systems, the first and second components of the equation are solved for U_j and V_j, and then the third and fourth components determine the corresponding functions P_j and W_j.

The first and second components of (3.12a) give two homogeneous equations for U_0 and V_0 that are consistent if

$$g_0 = \frac{(2s)^{\frac{1}{2}}}{V_{00}^+ V_{01}^+}, \tag{3.14}$$

which thus determines the first term in the expansion of the Görtler number. The functions U_0, P_0, W_0 can then be written in the form

$$U_0 = \frac{(2s)^{\frac{1}{2}} V_0}{V_{01}^+}, \quad W_0 = \frac{-i V_{0\eta}}{V_{01}^+}, \quad P_0 = \frac{V_{0\eta}}{V_{01}^+}, \tag{3.15}$$

where V_0 is to be determined at higher order. We now consider the solution of (3.12*b*), which can be evaluated by first of all eliminating U_1 from the first two component of this equation to give

$$\frac{g_1 V_{00}^+ V_{01}^+ V_0}{(2s)^{\frac{1}{2}}} + \frac{\eta V_0}{V_{00}^+ V_{01}^+} \{V_{01}^{+2} + 2V_{00}^+ V_{02}^+\} = 0. \tag{3.16}$$

We now use the fact that $\bar{V}_0(\xi)\,\bar{V}_0'(\xi)$ has a maximum at $\xi = \xi^+$, which means that the coefficient of ηV_0 in the above equation is zero, so that a consistent solution of the first two components of (3.12*b*) exists only if

$$g_1 = 0. \tag{3.17}$$

The solution of (3.12*b*) can then be written in the form

$$U_1 = \frac{(2s)^{\frac{1}{2}}}{V_{01}^+} V_1 + \frac{(2s)^{\frac{1}{2}}}{V_{00}^+} \eta V_0, \tag{3.18a}$$

$$W_1 = -\left\{\frac{i V_1}{V_{01}^+} + \frac{i\eta V_0}{V_{00}^+}\right\} \eta, \tag{3.18b}$$

$$P_1 = \left\{\frac{V_1}{V_{01}^+} + \frac{\eta V_0}{V_{00}^+}\right\} \eta, \tag{3.18c}$$

where V_1 is to be determined at higher order.

If we now consider the first two components of (3.12*c*) we find that these two inhomogeneous equations for U_2 and V_2 have a consistent solution only if

$$\frac{\partial^2 V_0}{\partial \eta^2} + 2s\eta^2 \left[\frac{V_{02}^+}{V_{00}^+} + \frac{V_{03}^+}{V_{01}^+}\right] V_0 + \tfrac{1}{3} g_2 V_{00}^+ V_{01}^+ \sqrt{2s}\, V_0 = 0. \tag{3.19}$$

If we now define the variable θ by

$$\theta = s^{\frac{1}{4}} \eta \left\{-8\left[\frac{V_{02}^+}{V_{00}^+} + \frac{V_{03}^+}{V_{01}^+}\right]\right\}^{\frac{1}{4}}, \tag{3.20}$$

and set

$$a = -\frac{g_2 V_{00}^+ V_{01}^+}{6\left[-\left(\dfrac{V_{02}^+}{V_{00}^+} + \dfrac{V_{03}^+}{V_{01}^+}\right)\right]^{\frac{1}{2}}}, \tag{3.21}$$

then we see that V_0 satisfies the parabolic-cylinder equation

$$\frac{\partial^2 V_0}{\partial \theta^2} - \tfrac{1}{4}\theta^2 V_0 - a V_0 = 0, \tag{3.22}$$

and if we impose the condition that the disturbance is confined to the $\epsilon^{\frac{1}{2}}$ layer we must choose $a = -\frac{1}{2} - m$, where m is a non-negative integer. This condition determines the infinite sequence of eigenvalues

$$g_2 = g_{2m} = \frac{6\{-(V_{02}^+/V_{00}^+ + V_{03}^+/V_{01}^+)\}^{\frac{1}{2}}\{\frac{1}{2} + m\}}{V_{00}^+ V_{01}^+}. \tag{3.23}$$

The most dangerous mode (corresponding to the minimum value of g_2) is clearly g_{20}. The eigenfunction corresponding to the eigenvalue g_{2m} is given by

$$V_0 = V_{0m}(\theta) = e^{-\frac{1}{4}\theta^2}\,\mathrm{He}_m(\theta), \tag{3.24}$$

where He_m is a Hermite polynomial. The solution of (3.12*c*) can then be written in the form

$$U_2 = g_0 V_{00}^+ V_2 + V_{0m}\eta^2\left\{-\frac{(2s)^{\frac{1}{2}}}{V_{01}^+}\right\}\left\{\frac{V_{02}^+}{V_{00}^+}+\frac{2V_{03}^+}{V_{01}^+}\right\}+\tfrac{1}{3}V_{00}^+ g_2 V_{0m}+\eta V_1 g_0 V_{01}^+, \tag{3.25a}$$

$$W_2 = \frac{-iU_{2\eta}}{(2s)^{\frac{1}{2}}}, \tag{3.25b}$$

$$P_2 = iW_2 - \frac{i}{2s}W_{0m\eta\eta}. \tag{3.25c}$$

We now consider the solution of (3.12*d*), which we determine by eliminating U_3 and V_3 between the first two components of this equation to give

$$\frac{\partial^2 V_1}{\partial\eta^2}+2s\eta^2\left[\frac{V_{02}^+}{V_{00}^+}+\frac{V_{03}^+}{V_{01}^+}\right]V_1+\tfrac{1}{3}g_2 V_{00}^+ V_{01}^+\sqrt{2s}\,V_1$$
$$= \{\tfrac{4}{3}\beta_3 s V_{00}^+ - V_{00}^+ V_{01}^+ g_3 \tfrac{1}{3}(2s)^{\frac{1}{2}}\}V_{0m}+S(\eta,s), \tag{3.26}$$

which, apart from the inhomogeneous terms on the right-hand side, is identical with (3.9). The function S is odd in θ if V_{0m} is even, and vice versa. In this case (3.26) can only have a solution if

$$g_3 = \frac{2(2s)^{\frac{1}{2}}}{V_{01}^+}\beta_3,$$

so that if we are interested in the neutral case then it follows from (3.10) that

$$g_3 = g_{3m} = \frac{\sqrt{2}s^{-\frac{1}{4}}}{V_{01}^+}\xi^+\left\{-8\left[\frac{V_{02}^+}{V_{00}^+}+\frac{V_{03}^+}{V_{01}^+}\right]\right\}^{\frac{1}{4}}\frac{V_{0m}'(\theta^*)}{V_{0m}(\theta^*)}, \tag{3.27}$$

where
$$\theta^* = s^{\frac{1}{2}}\eta^*\left\{-8\left[\frac{V_{02}^+}{V_{00}^+}+\frac{V_{03}^+}{V_{01}^+}\right]\right\}^{\frac{1}{4}}. \tag{3.28}$$

The procedure described above can in principle be continued to any order in ϵ, but we shall not do so here. We note that (3.27) gives the first non-parallel flow effect on the position where the Görtler vortex is neutrally stable. We further note that the above analysis can be done without first setting $\beta_0 = \beta_1 = \beta_2 = 0$. In such a case we would take the coefficients g_0, g_1 etc. to be given, and instead of obtaining eigenrelations of the form (3.23) we would obtain equations determining β_0, β_1 etc. in terms of g_0, g_1 etc. The expansion up to order $\epsilon^{\frac{3}{2}}$ in G that we have just derived is valid for $\delta^{-2} \ll Re \ll \delta^{-\frac{26}{9}}$. The upper limit of this range increases if fewer terms in the expression of G are retained.

4. Discussion of results

Suppose that the quantities V_{00}^+, V_{01}^+ etc. that appear in (3.14), (3.23) and (3.27) are replaced by their numerical values obtained by integrating the nonlinear equation for $f(\xi)$. We find that the mth linear mode with wavenumber ϵ^{-1} is neutral at the location s if the Görtler number G is given by

$$G = G^{(m)} = \frac{5{\cdot}91}{\epsilon^4}\left\{s^{\frac{1}{2}}+0{\cdot}96[1+2m]\epsilon+\frac{1{\cdot}16\,V_{0m}'(\theta^*)\,s^{-\frac{1}{4}}\epsilon^{\frac{3}{2}}+\dots}{V_{0m}(\theta^*)}\right\}, \tag{4.1}$$

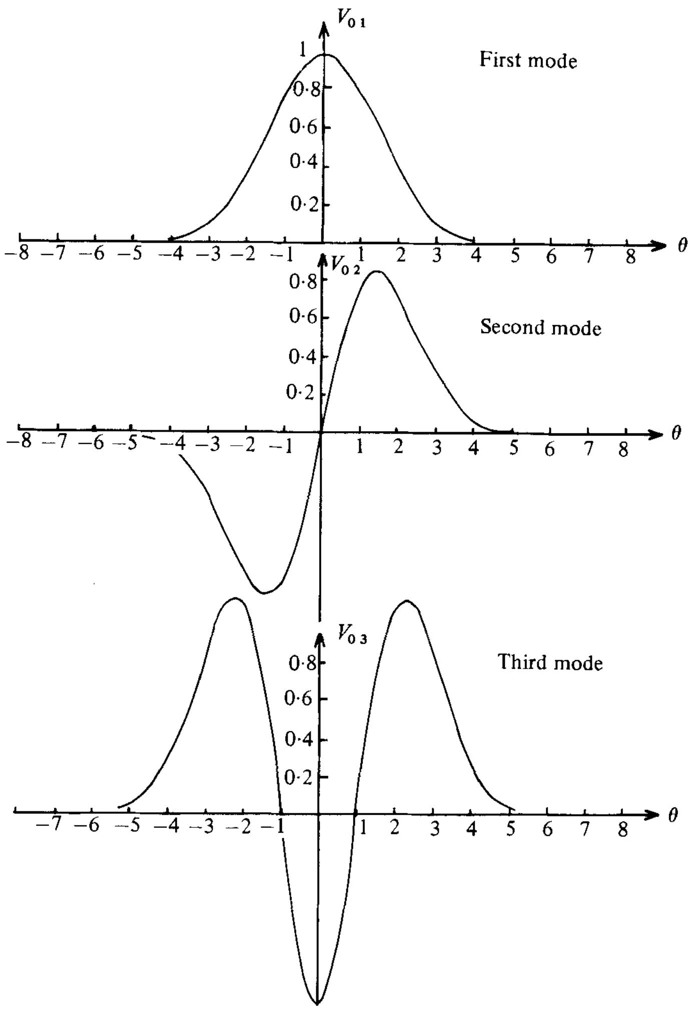

FIGURE 2. The first three disturbance eigenfunctions for the Görtler problem.

and the most dangerous disturbance clearly corresponds to $m = 0$. In figure 2 we have shown the first three eigenfunctions $V_{00}(\theta)$, $V_{01}(\theta)$ and $V_{02}(\theta)$, and we see that the number of vortices associated with each mode increases with m. In fact the asymptotic analysis given in §3 becomes invalid when $m = O(\epsilon^{-1})$, and the WKB method must be used to describe such modes. These modes have critical layers of thickness $\epsilon^{\frac{2}{3}}$ located $O(1)$ distances either side of $\xi = \xi^+$. The solutions in these layers are in terms of the Airy function Ai and in between the layers there is a third region where the solution is almost periodic, but smaller in magnitude by $O(\epsilon^{\frac{1}{6}})$. Above the upper critical layer and below the lower critical layer the disturbance is exponentially small, so that the disturbance is again trapped in the interior of the fluid.

We shall now concentrate on the most dangerous linear mode and introduce a local Görtler number G_s defined by

$$G_s = 2\left(\frac{V_0^* ls}{\nu}\right)^{\frac{1}{2}} \frac{ls}{R} = Gs^{\frac{3}{2}}. \tag{4.2}$$

If we then define the local wavenumber ϵ_s^{-1} by

$$\epsilon_s = \epsilon s^{-\frac{1}{2}}, \tag{4.3}$$

then we see that (4.1) with $m = 0$ can be written in the form

$$G_s^{(0)} = \frac{5{\cdot}91}{\epsilon_s^4}\{1+0{\cdot}96\epsilon_s - 0{\cdot}58\theta^*\epsilon_s^{\frac{3}{2}} + \ldots\}. \tag{4.4}$$

Here we have used the fact that $V'_{00}/V_{00} = -\frac{1}{2}\theta^*$. We see that the non-parallel-flow effect, manifested through the $O(\epsilon_s^{\frac{3}{2}})$ term, can either increase or decrease $G_s^{(0)}$, depending on the sign of θ^*. If measurements are made below the critical level $\theta^* = 0$, then the correction term is positive and $G_s^{(0)}$ is greater than the value given by the parallel-flow approximation, which retains only the first two terms in (4.4). We note that, if a flow quantity other than the normal velocity component were used, then a slightly different form for the $O(\epsilon_s^{\frac{3}{2}})$ term in (4.4) would be found. However the z-velocity component and the pressure have zeros at $\theta^* = 0$, so that the relative change of these quantities would become infinite there, and so we do not consider stability criteria based on these quantities. The downstream velocity component leads to a value of $G_s^{(0)}$ identical to (4.4) if it is taken as a representative flow quantity.

We notice in (4.4) that the order $\epsilon_s^{\frac{3}{2}}$ term becomes increasingly important for large values of θ^*, so that at the edge of the critical layer, where $\theta^* \sim \epsilon_s^{-\frac{1}{2}}$, the second and third terms are comparable. However, if measurements are made at the centre of the critical layer the $O(\epsilon_s^{\frac{3}{2}})$ term is zero, and the first non-parallel flow effect appears at order ϵ_s^2. Thus, depending on where in the flow field the local stability criterion is used, the non-parallel-flow effect in (4.4) can vary by $O(\epsilon_s)$. It would seem sensible to perform experiments with $\theta^* = 0$ so that non-parallel-flow effects are in some sense minimized.

The above discussion explains why it is not easy to interpret the available experimental results, which are due to Tani & Sakagami (1962) and Bippes & Görtler (1972). It is not clear from either of these papers what were the appropriate stability criteria used by the respective authors. However, Bippes & Görtler measured the three components of the disturbance velocity field, so we can at least see if our disturbances have the correct shape. In figure 3 we have compared our results with those of Bippes & Görtler for $G_s = 157{\cdot}3$ and $ls/R = 0{\cdot}475$. The value of ϵ_s corresponding to this value of G_s can be evaluated approximately by retaining only the first term in (4.4), to give $\epsilon_s = 0{\cdot}66$. The theoretical and experimental results have both been normalized to give a downstream velocity component with maximum value 1. The agreement between the results seems reasonable except perhaps for the discrepancy between the measured and predicted z-velocity components. This discrepancy is possibly due to finite-amplitude effects and/or the size of ϵ_s^{-1} being too small to allow the first term in (3.8) to describe the flow field accurately. Nevertheless, the similarity between the observations and our theoretical predictions encourages us to believe that the asymptotic solution has some application in regime of physical interest.

In figure 4 we have shown the dependence of $G_s^{(0)}$ on ϵ_s for $\theta^* = 0$ and $\theta^* = 1$. The latter value corresponds to the local stability criterion being applied below the critical layer. The difference between these curves demonstrates the non-parallel-flow effect on the Görtler number. In figure 5 we have shown the neutral curves predicted by (4.4) at relatively low values of ϵ_s^{-1}. The three theoretical curves shown correspond to taking $\theta^* = 0$, -2, and -4 in (4.4). The experimental results of Bippes & Görtler and Tani & Sakagami lie in the shaded region of this figure. We see that the neutral curves corresponding to θ^* decreasing move closer to the experimental results. In the absence of any knowledge of the stability criterion used by the above authors, the significance of this result is not clear.

In figure 5 we have also shown the theoretical results of Görtler (1940) and Smith

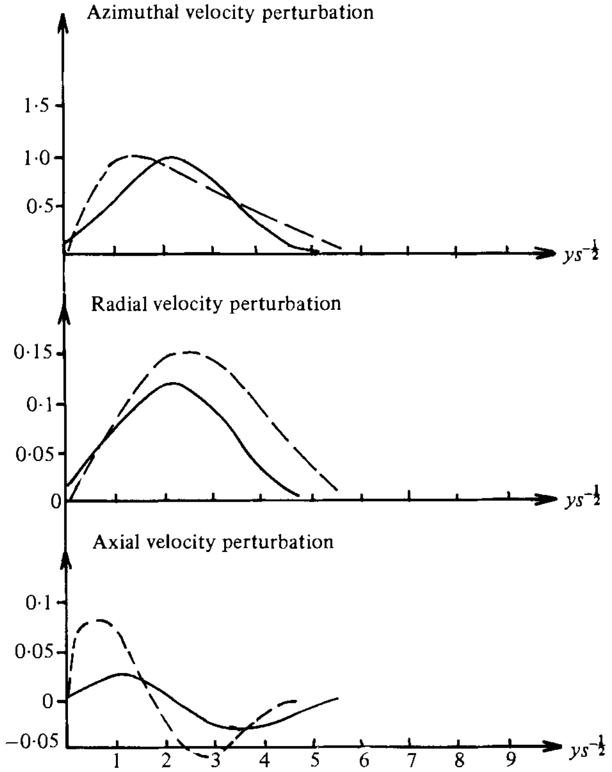

FIGURE 3. A comparison between the experimental (– – –) and theoretical (——) eigenfunctions.

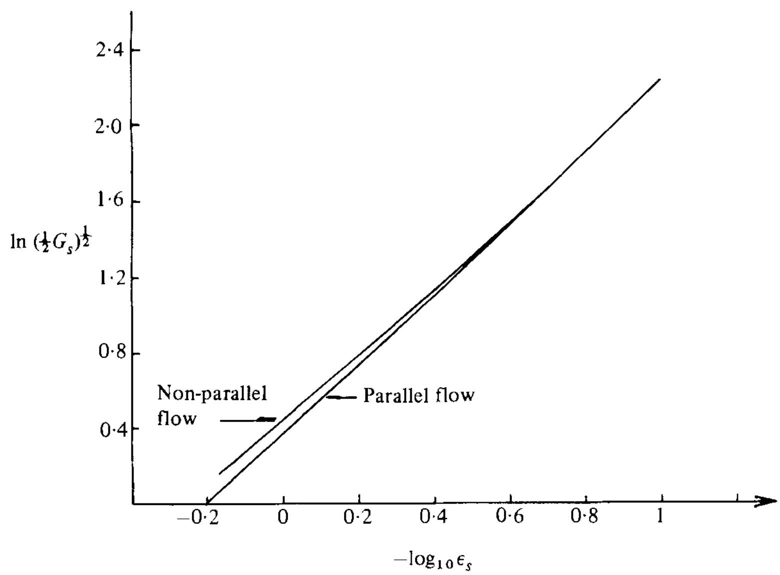

FIGURE 4. The parallel and non-parallel neutral curves.

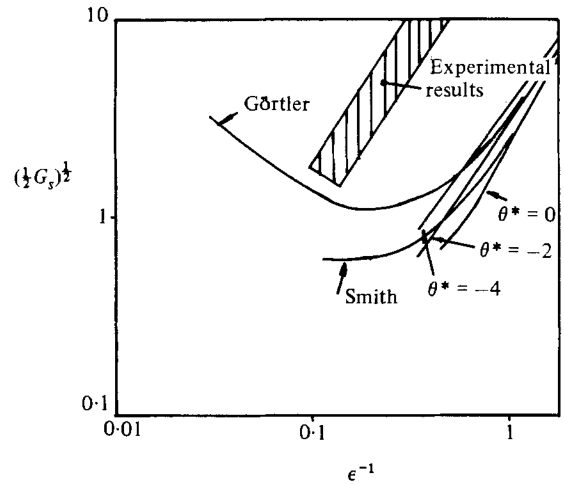

FIGURE 5. A comparison with previous theoretical and experimental work.

(1955). The latter authors were concerned with external-flow boundary layers, but the lowest order terms in this case are identical with those given in (3.4). Görtler was interested primarily in the regions $G \sim O(1)$ and solved the stability problem obtained by ignoring all the terms on the right-hand sides of (3.4). Smith (1955) solved similar equations but included some of the terms on the right-hand sides of (3.4). Neither set of equations corresponds to a rational approximation to (3.4) if $G \sim O(1)$. The equations solved by Görtler were later solved more accurately by Hämmerlin (1955), and in a subsequent paper Hämmerlin (1956) solved the equations obtained by retaining some of the higher-order curvature terms in (3.4). Clearly none of these procedures can be justified, but we note that the neutral curves of Smith and Görtler lie close to our curves for sufficiently small values of ϵ_s. This is not surprising, since in this case the equations solved by Görtler and Smith give the first two terms in an expansion of the Görtler number in terms of ϵ_s identical with the first two terms of (4.4). However, neither approximate set of equations leads to a term similar to the order $\epsilon_s^{\frac{3}{2}}$ term in (4.4). It would therefore seem that the present asymptotic formulation of the problem provides the first rational framework for investigating the effect of boundary-layer growth on Görtler vortices. The similarity of our approach to that used by Smith (1979, 1980) to investigate Tollmien–Schlichting waves in growing boundary layers is worth noticing. We further mention that several authors apart from those discussed above have solved various reduced forms of (3.4) with the aim of obtaining a neutral curve having a critical wavenumber at a finite value of ϵ_s. Without exception, these reduced equations do not take account of boundary-layer growth and therefore cannot be justified. It seems likely that the problem of determining what happens to the neutral curve when $\epsilon_s^{-1} \to 0$ can only be resolved by a numerical investigation of the partial differential equations (3.4).

We now turn to the question of how the vortices develop into finite-amplitude motions. Bippes & Görtler (1972) have observed experimentally that further downstream from the location where the vortices are first observed the flow becomes time-dependent. Moreover, the vortices then become almost periodic in nature and are very similar in appearance to wavy-vortex flows observed between concentric cylinders. The 'waviness' of the vortices appears to increase with s, and ultimately the flow becomes turbulent. We expect that the nonlinear stage during which the wavy vortices appear must be described using a weakly nonlinear theory of the type used so successfully for the Taylor problem by Davey, DiPrima & Stuart (1968). We do not pursue such a calculation here, but at least make a first step towards understanding the wavy-vortex regime by determining whether the disturbances investigated in §3 can be modified such that the Görtler-vortex structure propagates downstream with some speed. It seems that such three-dimensional modes have not been considered previously, and there is no obvious reason why they should be more stable than the modes considered in §3. However, on the basis of the experimental results of Bippes & Görtler (1972) and the experience of previous investigators of the Taylor problem we expect that this is not the case.

The first step in investigating such modes is to decide on appropriate scales for the s- and t-dependence to be introduced into (3.7*b*). If the s- and t-dependences are to balance in some sense, then from (3.4) we require that $\partial/\partial t \sim \partial/\partial s$. After some trial and error we find that the most rapid variation with s that can be incorporated into the expansion procedure of §3 is such that $\partial/\partial s \sim \epsilon^{-\frac{3}{2}}$. We therefore define a new variable S by

$$S = \epsilon^{-\frac{3}{2}}s,$$

and the appropriate time variable T is defined by

$$T = \epsilon^{-\frac{3}{2}}t.$$

We now look for perturbations to the basic flow of the form (3.8) but with iz/ϵ in (3.7*b*) replaced by $i\{z/\epsilon + kS - \Omega T\}$. We again assume for simplicity that $\beta_0 = \beta_1 = \beta_2 = 0$, and find the Görtler number G at which a disturbance with fixed transverse and downstream wavenumbers ϵ^{-1} and $k\epsilon^{-\frac{3}{2}}$ and fixed frequency $\Omega\epsilon^{-\frac{3}{2}}$ is neutrally stable. The expansion (3.8) is retained, and the procedure of §3 is repeated. At first order we find that g_0 is again given by (3.14), while at order $\epsilon^{\frac{1}{2}}$ we find that $g_1 = 0$ and that the constants k and Ω must satisfy the equation

$$V_{00}^{+} k = \Omega,$$

so that the three-dimensional modes propagate downstream with the speed of the fluid of the basic state in the critical layer. Thus for the Blasius boundary layer the wave propagates downstream with a speed equal to 0·67 times that of the free-stream speed. At order ϵ we find that V_0, the first-order velocity component in the s-direction, satisfies the equation

$$\frac{\partial^2 V_0}{\partial \eta^2} + 2s\eta^2\left[\frac{V_{02}^{+}}{V_{00}^{+}} + \frac{V_{03}^{+}}{V_{01}^{+}}\right]V_0 + \tfrac{1}{3}g_2 V_{00}^{+} V_{01}^{+}(2s)^{\frac{1}{2}} - \tfrac{4}{3}ik\eta s V_{01}^{+} V_0 = 0,$$

and this equation must again be solved subject to the condition $V_0 \to 0$, $|\eta| \to \infty$. If we now define the variable η' by

$$\eta' = \eta - ikV_{01}^{+}/3\left\{\frac{V_{02}^{+}}{V_{00}^{+}} + \frac{V_{03}^{+}}{V_{01}^{+}}\right\},$$

then we find that V_0 satisfies (3.19) but with η replaced by η' and g_2 by

$$g_2 + \frac{(2s)^{\frac{1}{2}}k^2 V_{01}^{+}}{3V_{00}^{+}}\left[\frac{V_{02}^{+}}{V_{00}^{+}} + \frac{V_{03}^{+}}{V_{01}^{+}}\right]^{-1}.$$

Thus the disturbance with transverse wavenumber ϵ^{-1}, downstream wavenumber $k\epsilon^{-\frac{3}{2}}$ and frequency $\Omega\epsilon^{-\frac{3}{2}}$ is neutrally stable at the location s if

$$G = G_k^{(m)} = \frac{5{\cdot}91}{\epsilon^4}\{s^{\frac{1}{2}} + \epsilon[0{\cdot}96(1+2m) + 0{\cdot}21s^{\frac{1}{2}}k^2] + O(\epsilon^{\frac{3}{2}})\},$$

so that the travelling-wave disturbances are more stable than the stationary disturbances considered in §3.

Finally we note that the discussion in §3 is easily modified to allow for the concave surface having curvature $K(s)$ dependent on s. The only essential difference in this case is that the coefficient of G in (3.4) is then multiplied by $K(s)$. The subsequent analysis is unchanged except that G is then replaced by $GK^{-1}(s)$.

The author acknowledges the useful comments on a first draft of this paper made by J. T. Stuart. The author is also grateful to the referee who supplied the numerical results described in §2 and for his other constructive comments in connection with §2.

REFERENCES

BIPPES, H. & GÖRTLER, H. 1972 *Acta Mech.* **14**, 251.

CHANDRASEKHAR, S. 1958 *Hydrodynamic and Hydromagnetic Stability*. Clarendon.

DAVEY, A., DIPRIMA, R. C. & STUART, J. T. 1969 *J. Fluid Mech.* **31**, 17.

DEAN, W. R. 1928 *Proc. R. Soc. Lond.* A **121**, 402.

FLORYAN, J. & SARIC, W. 1979 *A.I.A.A. Paper* no. 79–1497.

GASTER, M. 1974 *J. Fluid Mech.* **64**, 465.

GÖRTLER, H. 1940 *NACA Tech. Memo.* no. 1357.

GREGORY, N. & WALKER, W. S. 1950 *A.R.C. R. & M.* no. 2779.

HÄMMERLIN, G. 1955 *J. Rat. Mech. Anal.* **4**, 279.

HÄMMERLIN, G. 1956 *Z. angew. Math. Phys.* **7**, 156.

HERBERT, T. 1976 *Arch. Mech.* **28**, 1039.

MEKSYN, D. 1946 *Proc. R. Soc. Lond.* A **187**, 115.

SMITH, A. M. O. 1955 *Appl. Math.* **13**, 233.

SMITH, F. T. 1979 *Proc. R. Soc. Lond.* A **366**, 91.

SMITH, F. T. 1980 *Proc. R. Soc. Lond.* A **368**, 573.

STUART, J. T. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead), chap. X. Oxford University Press.

TANI, I. & SAKAGAMI 1962 *J. Proc. Int. Council Aero. Sci. Third Congress*, p. 391.

TAYLOR, G. I. 1923 *Phil. Trans. R. Soc. Lond.* A **223**, 289.